

Differentiation and Integration Story

What differentiation and integration do is compute rates of change and areas/volumes under a curve, respectively. This is found everywhere in the natural sciences and engineering. Many of these problems nowadays can be solved by discrete approximations using computers, but the algorithms are still built using the theory of differentiation and integration.

Let us go over some common problems and see how differentiation and integration help.

Differentiation and integration can be used to build (and solve) differential equations. The two sorts of big divisions in differential equations are ordinary and partial differential equations.

Ordinary differential equations are typically studied in the form of dynamical systems, applications include population dynamics, modelling of epidemics, neural networks, robotics, and cognitive science.

Partial differential equations can be used to model flow of fluids (Navier-Stokes equations), space-time and gravity (Einstein field equations), electromagnetism (Maxwell's equations), in quantum mechanics (Schrodinger equation, Klein-Gordon equation, Dirac equation, etc. etc. etc.).

Differentiation is fundamental to something called the Newton-Raphson method, which allows us to find roots of nonlinear equations. Again, a problem of interest in sciences and engineering.

The concepts are also fundamental to numerical analysis, making discrete versions of above-mentioned equations to come up with solutions using computers.

The above statements are somewhat broad and cursory, but it is a rather broad question, and doing any of these topics justice usually requires a full university level course.

Differential equations (DE), which are equations that relate functions with their derivatives, and there are analytic methods to solve that use integration, as well as numerical integrators, looks like:

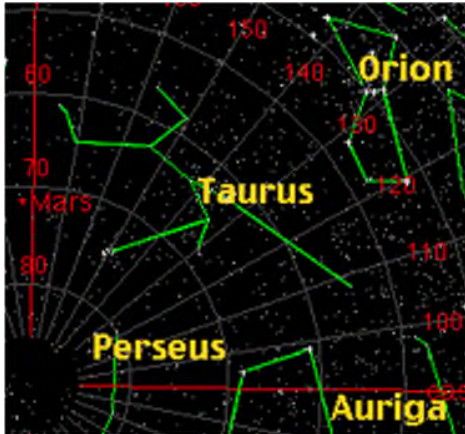
$$d^2u/dx^2 + \omega^2 u = 0$$

DEs are everywhere in our lives. Light can be described by a wave equation, and similarly quantum particles (in your computer, for instance) are also described by a [slightly different] wave equation. Anywhere where there is water flowing can be described by a DE.

Aerodynamics, vibrations, propulsion, electronics, sprinklers, traffic jams, population growth and decay, image processing, machine vision, neural networks, weather, heat transfer, engine efficiency, climate change, structural integrity, nuclear weapons, artillery trajectory, solar cells, financial derivatives pricing, and even the coffee cooling in your cup are all described by differential equations.

Differential equations are everywhere, and thus calculus is everywhere too.

In Isaac Newton's day, one of the biggest problems was poor navigation at sea.



Before calculus was developed, the stars were vital for navigation.

Shipwrecks occurred because the ship was not where the captain thought it should be. There was not a good enough understanding of how the Earth, stars and planets moved with respect to each other.

Calculus (differentiation and integration) was developed to improve this understanding.

Differentiation and integration can help us solve many types of **real-world problems**. We use the **derivative** to determine the **maximum and minimum values** of particular functions (e.g. cost, strength, amount of material used in a building, profit, loss, etc.).

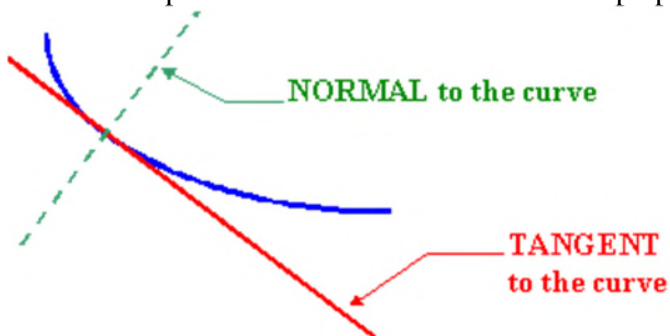
Derivatives are met in many **engineering and science problems**, especially when modelling the behaviour of moving objects.

Let the discussion begin with some general applications which we can then apply to specific problems.

Tangents and Normals

(Tangents and normals to curves when we are analysing forces acting on a moving body such as forces on a car turning a corner)

A **tangent** to a curve is a line that touches the curve at one point and has the same **slope** as the curve at that point. A **normal** to a curve is a line perpendicular to a tangent to the curve.



If we are traveling in a car around a corner and we hit something slippery on the road (like oil, ice, water or loose gravel) and our car starts to skid, it will continue in a direction **tangent** to the curve. Likewise, if we hold a ball and swing it around in a circular motion then let go, it will fly off in a **tangent** to the circle of motion.

When going fast around a circular track in a car, the force that one feels, pushing one outward, is **normal** to the curve of the road. Interestingly, the force that is making you go around that corner is actually directed towards the **center** of the circle, normal to the circle. The spokes of a wheel are placed **normal** to the circular shape of the wheel at each point where the spoke connects with the center.

Example of a curve: $y = x^3 - 2x^2 + 5$; Therefore, the gradient or the slope is $dy/dx = 3x^2 - 4x$

Differentiation example in MATLAB (syms construct defines symbolic variables):

```
>> syms x
>> f(x) = x^3 - 2*x^2 + 5;
>> d1fx = diff(f,1)
```

$d1fx(x) = 3*x^2 - 4*x$

Integration example in MATLAB:

```
>> intf = int(f)
```

$intf(x) = (x*(3*x^3 - 8*x^2 + 60))/12$

A structural engineer must choose the proper beam size to build a bridge. How do they know how much a beam will deflect under load? Well normally, the engineer will just refer to a book of tables, tabulating the deflection for any given beam, but those tables were built using differential equation models to predict the behaviour of beams.

How about integration? Integration tells us an area, perhaps a volume. For example, an engineer might compute the volume of a cylindrical tank (with spherical end caps) used to store water. To compute that volume, those curved sides will involve integration.

As it was mentioned earlier, in Isaac Newton's time, one of the biggest problems was poor navigation at sea. Before calculus was developed, the stars were vital for navigation. Shipwrecks occurred because the ship was not where the captain thought it should be. There was not a good enough understanding of how the Earth, stars and planets moved with respect to each other.

Calculus (differentiation and integration) was developed to improve this understanding.

Calculus is a branch of mathematics that helps us understand changes between values that are related by a function. Calculus is used in many different areas such as physics, astronomy,

biology, engineering, economics, medicine and sociology.

Differentiation and integration can help us solve many types of real-world problems.

We use the derivative to determine the maximum and minimum values of particular functions (e.g. cost, strength, amount of material used in a building, profit, loss, etc.). Derivatives are met in many engineering and science problems, especially when modeling the behaviour of moving objects.

Examples of some general area of applications of calculus are:

1. Economics for functional relationships involving dependent and independent variables
2. History for predicting the life of a stone
3. Geography for study of gases present in the atmosphere
4. Aviation for pilots to measure the air pressure

We use the derivative to determine the maximum and minimum values of particular functions (e.g. cost, strength, amount of material used in a building, profit, loss, etc.).

Integration enables find displacement (from velocity) and velocity (from acceleration) using the indefinite integral.

Some examples of the use of integration are:

1. Calculating area under a curve
2. Calculating area in between the two curves
3. Determining volume of a solid object
4. Finding centre of mass for an object with curved sides
5. Determining moments of inertia for find the resistance of a rotating body
6. We use integration when the shape has curved sides
7. Determining amount of work by a variable force
8. Calculating the work done when electric charges are separated.

More Practice Problems

Example 1

The distance moved by a mass oscillating on a spring is given by equation:

$$x=5\cos(8t) \text{ mm}$$

Find the distance and velocity after 0.1 seconds

Solution:

$$\text{At } 0.1 \text{ seconds } x=5\cos(.8t)=3.48 \text{ mm}$$

$$\text{velocity, } v=dx/dt=-40\sin(8t)=-40\sin(.8)=-28.69/s$$

In MATLAB

```
>> syms t
>> f(t)= 5*cos(8*t);
>> x = f('1')
x = 3.48
>> df1d = diff(f,1)
```

```
df1d(t) = -40*sin(8*t)
>> v = df1d('.1')
v = -28.69
```

Example 2

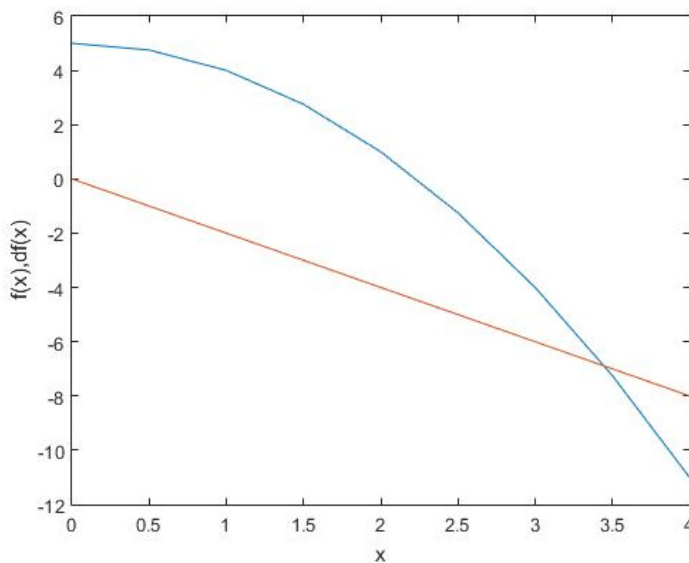
The given parabolic curve is:

$$y = -x^2 + 5$$

Find the slope (derivative) of the parabola dy/dx

Plot both y and dy/dx .

```
>> syms x
>> f(x) = -x^2 + 5;
>> d1f= diff(f,1);
d1f(x) = -2*x
>> x = 0:.5:4;
>> plot(x,f(x))
>> hold on
>> plot(x,d1f(x))
>> xlabel('x')
>> ylabel('f(x),df(x)')
```



Example 3

G354: The amount of heat Q needed to increase the temperature of a unit of mass of water from 0^0 C to t^0 C is given the formula:

$$Q = t + 0.00002t^2 + 0.0000003t^3 \text{ cal/kg}$$

Find the specific heat of water at $t = 100^0$ C.

The specific heat:

$$c(t) = dQ/dt$$

$$c(t) = 1 + 0.00004t + 0.0000009t^2$$

For $t = 100^0$ C

$$c_{100} = 1 + 0.004 + 0.009 = 1.013$$

In MATLAB

```
>> syms t
>> Q(t)=t+.00004*t^2+.0000003*t^3;
>> c=diff(Q,1);
>> c100=c('100');
>> display(double(c100))
ans = 1.0170
```

Example 4

The force between charges is proportional to the product of their charges and inversely proportional to the square of the distance between them.

Therefore,

$$f(x)=k q_1 q_2 / x^2$$

where q_1 and q_2 are expressed in units of coulombs, C

Work done when electric charges move toward each other (or when they are separated) is given by:

$$\text{work} = \int q_1 q_2 x^{-2} dx$$

An electron has a 1.6×10^{-19} C negative charge.

Find the amount of work in the interval from 1 to 4 pm, where pm stands for picometre or 10^{-12} metre

The given interval of integration are:

$$a = 1 \times 10^{-12} \text{ m}$$

$$b = 4 \times 10^{-12} \text{ m}$$

The of k is:

$$k = 9 \times 10^9$$

The values of q_1 and q_2 are:

$$q_1 = q_2 = 1.6 \times 10^{-19} \text{ C}$$

Therefore:

Work = $\int q_1 q_2 x^{-2} dx$ in the interval from a to b.

$$= \int (9 \times 10^9) * (-1.6 \times 10^{-19})^2 / x^2 dx$$

$$= (2.304 \times 10^{-28}) [-1/x]$$

$$= 1.728 \times 10^{-16} \text{ J}$$

In MATLAB:

```
>> k=9*10^9;
>> q1=1.6*10^-19;
>> q2=q1;
>> syms x
>> f(x)=k*q1*q2/x^2;
```

```
>> w=int(f,1*10^-12,4*10^-12);
>> display('work:'), display(double(w))
work: 1.7280e-16
```

Example 5: Calculus Mean Value Theorem of Differentiation Role in Catching Speeders

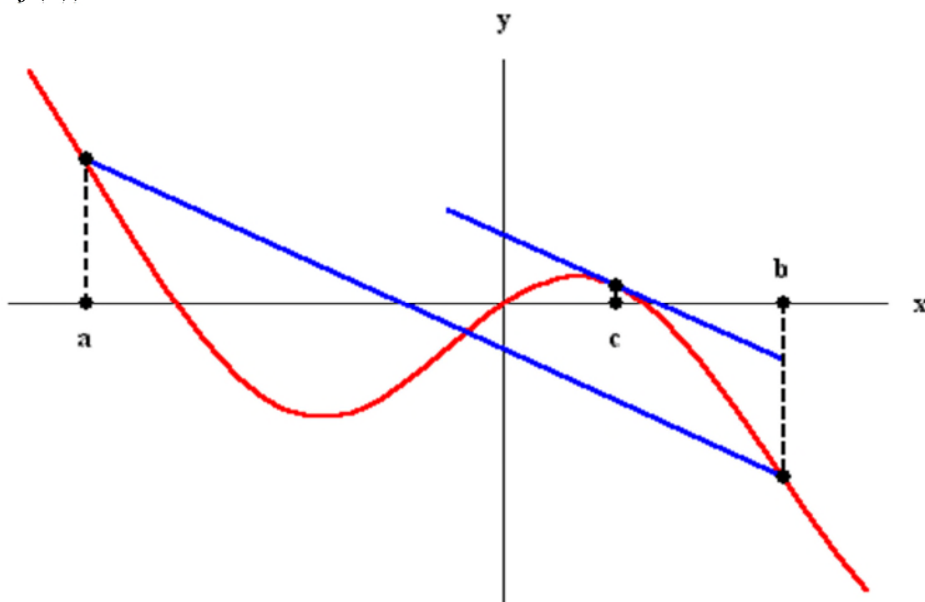
Usually there are police cars on the side of a road with their radar guns turned on for catching speeders. Speeders with a sharp eye slow down whenever they spot a police car and thus avoid getting caught. Police found that by using differentiation they can catch the cheaters even if they slow on spotting a car.

The famous Mean Value Theorem comes to the rescue for police.

The **Mean Value Theorem** is one of the most important theoretical tools in Calculus. It states that if $f(x)$ is defined and continuous on the interval $[a,b]$ and differentiable on (a,b) , then there is at least one number c in the interval (a,b) (that is $a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

where $f'(c)$ is the slope of the line passing through $(a,f(a))$ and $(b,f(b))$. conclusion of So the conclusion of the Mean Value Theorem states that there exists a point c within (a,b) such that the tangent line is parallel to the line passing through $(a,f(a))$ and $(b,f(b))$ as shown below:



How does this help the police? Imagine that the cop I see off in a distance clocks the driver that just passed me. However, the driver saw the police car and slowed down to the speed limit of 50 mph in time, and avoided getting caught. But pretend now that one mile down the road and 50 seconds later, another police car clocks the same driver going 45 mph. Although the driver was not found to be speeding at both of the instants the two cops measured his speed, if these two cops compared their notes, and used the Mean Value Theorem, they would conclude that some point 'c' inside the interval $0 < t < 50$ the driver's speed, the velocity, $s'(t)$ must have been equal to

$$s'(c) = [s(50) - s(0)] / (50 - 0) = 1 \text{ mile} / 50 \text{ seconds} = 72 \text{ mph}$$

On this 50mph road, that would definitely qualify as speeding.

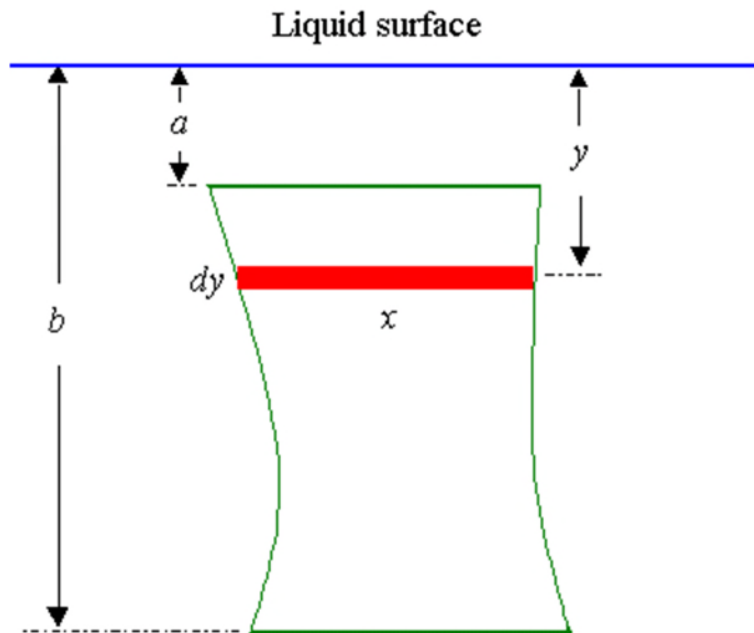
Example 6: Find the force on one side of a cubical container 6.0 cm on an edge if the container is filled with mercury. The weight density of mercury is 133 kN/m^3 .

The force F on an area A at a depth y in a liquid of density w is given by

$$F = wyA$$

The force will increase if the density increases, or if the depth increases or if the area increases. So, if we have an unevenly shaped plate submerged vertically in a liquid, the force on it will increase with depth. Also, if the shape of the plate changes as we go deeper, we have to allow for this.

So, we have:



Now, the total force on the plate is given by

$$F = w \int_a^b x y dy$$

where

x is the length (in m) of the element of area (expressed in terms of y)

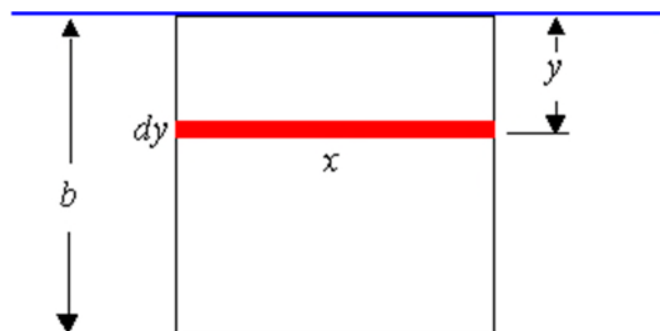
y is the depth (in m) of the element of area

w is the density of the liquid (in N m^{-3}), (for water, this is $w = 9800 \text{ N m}^{-3}$)

a is the depth at the top of the area in question (in m)

b is the depth at the bottom of the area in question (in m)

Example 1



This is a very basic example where the width of the plate does not change as we move down the plate.

It is always $x=6$.

Also, the depth of the top of the plate is 0, so $a=0$.

To apply the formula, we have:

$$x=0.06 \text{ m}$$

y is the variable of integration

$$w=133 \text{ kN m}^{-3}=133,000 \text{ N m}^{-3}; a=0; b=0.06$$

So we have:

$$\begin{aligned} \text{Force} &= w \int xy \, dy \text{ in the interval from } a \text{ to } b \\ &= 133000 \int_0^{0.06} y \, dy \\ &= 7980 \int_0^{0.06} y \, dy \\ &= 7980 \left[\frac{y^2}{2} \right] \text{ from } 0 \text{ to } .06 \\ &= 14.4 \text{ N} \end{aligned}$$

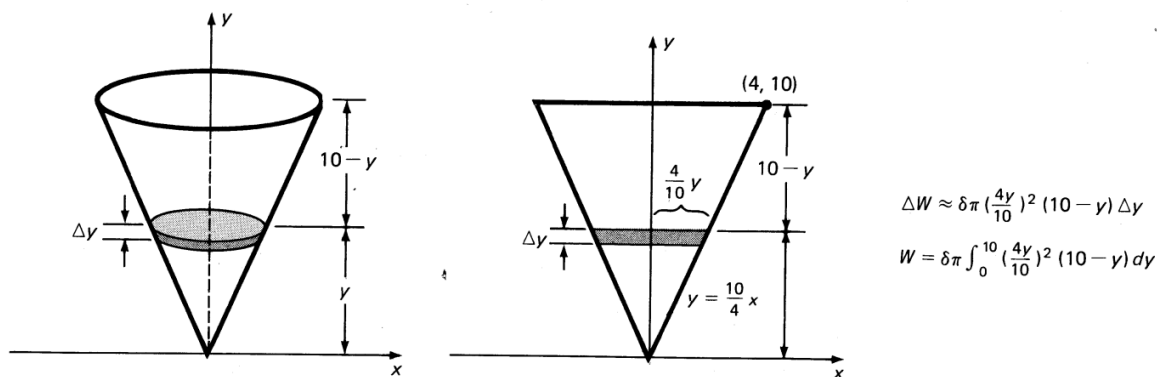
In MATLAB:

```
>> syms y
>> F(y)=7980*y;
>> FABINT(y)=int(F(y))
FABINT(y) = 3990*y^2
>> FABINT06=FABINT('0.06')
FABINT06 = 14.364
```

Example 7

Practical Integration Problem

A tank in the shape of a right circular cone is full of water. If the height of the tank is 10 feet and the radius of its top is 4 feet, find the work done in pumping the water over the top of the tank.



The coordinates system for the tank are shown above, in 3D view and in 2D cross section.

Imagine slicing the water into thin horizontal disks, each of which must be lifted over the edge of the tank. A disk of thickness Δy at height y has radius $4y/10$. Thus its volume is approximately $\pi(4y/10)^2\Delta y$ and its weight is about $\delta\pi(4y/10)^2\Delta y$, where $\delta=62.4$ is the (weight) density of water in pounds per cubic foot. The force necessary to lift this disk of water is its weight, and it must be lifted a distance of $10-y$. Thus, the work Δw done on this disk is approximately

$$\Delta w = (\text{force}) \cdot (\text{distance}) \approx \delta \pi (4y/10)^2 \Delta y \cdot (10-y)$$

Therefore,

$$w = \int \delta \pi (4y/10)^2 \Delta y \cdot (10-y) \text{ in the interval from } 0 \text{ to } 10$$

$$\begin{aligned}
&= \partial \pi(4/25) \int (10y^2 - y^3) dy \\
&= (4 \pi)(62.4)/25 [10y^3 - y^4/4] \\
&= 26,138 \text{ foot-pounds}
\end{aligned}$$

In MATLAB

```

>> syms y
>> delw(y)=62.4*pi*(4*y/10)^2*(10-y)
>> w(y)=int(delw(y))
w(y) =
-(104*pi*y^3*(3*y - 40))/125
>> w10=w('10')
w10 =
8320*pi
>> w10=8320*3.1416
w10 =
2.6138e+04
26138

```

Example 8: Calculating work done involving force and distance of movement

The force required to extend (or compress) an elastic spring x units longer (or shorter) than its natural length is proportional to x (at least for sufficiently small values of x). This is known as Hooke's law.

The force, $F(x)=kx$ where k is the spring constant for the particular spring.

If a force of 10 kg is required to extend a certain spring to 4 cm longer than its natural length, how much work must be done to extend it that far, given that $k=2.5$ kg/cm

$$\begin{aligned}
F(x) &= kx, F(4) = 2.5 \cdot 4 = 10 \\
W(x) &= \int kx dx \text{ in the interval from 0 to 4} \\
&= kx^2/2 \\
&= 2.5 \cdot 4^2/2 \\
&= 20 \text{ kg-cm}
\end{aligned}$$

In MATLAB

```

>> syms x
>> F(x)=2.5*x;
>> F4=F('4')
F4 = 10
>> W(x)=int(F(x));
>> W4=W('4')
W4 = 20

```

Example 9:

The gravitational force attracting a mass m located a height h above the surface of the earth is given by $F(h)=km/(r+h)^2$ where r is the radius of the earth and k is a constant independent of m

and h . Determine, in terms of k and r , the work that must be done against gravity to raise an object from the surface of the earth to (a) a height H above the surface of the earth, (b) an infinite height above the surface of the earth. Given that $r=6400$ km, $g=9.8$ m/sec²= $.0098$ km/sec²

According to the law of conservation of energy, the projectile must have sufficient kinetic energy at firing to do work necessary to raise the mass m to infinite height (that is, to supply sufficient potential energy to raise it to that height). This is km/r and $F_0=km/r^2$ as shown below.

$$F(h)=km/(r+h)^2, \text{ at } h=0, F_0=km/r^2$$

Let the mass of the object be 10 kgm, $k=gr^2=.0098(6400)^2$

$$dW= km/(r+h)^2 dh$$

The total work to raise it from height $h=0$ to $h=H$ is

$$W(h)= \int km/(r+h)^2 dh \text{ in the interval from } 0 \text{ to } H$$

$$=-km/(r+h)$$

$$= km/r \text{ for } h=0$$

According to Newton's second law of motion, $F=mg$ and $k=gr^2$

Therefore,

$$km/r^2 = mg$$

$$k=gr^2$$

If the initial velocity of the projectile is v , we want

$$\frac{1}{2}mv^2 \geq km/r$$

$$v \geq \sqrt{2k/r}$$

$$\geq \sqrt{2gr}$$

$$\geq \sqrt{2(.0098)(6400)}$$

$$\geq 11.2 \text{ km/sec}$$

In MATLAB

```
>> m=10;
>> r=6400;
>> g=.0098;
>> k=g*r^2;
>> syms h
>> F(h)=k*m/(r+h)^2;
>> W(h)=int(F(h));
>> W0=W('0')
W0 = -3136/5
>> v=(2*k/r)^.5
v = 11.2000
```

Example 10, Arch Length

Determine the length of the function of the graph of $f(x)=x^2$ in the interval 1 to 3.

Length of a function is given by

$$L(x)= \int \sqrt{1+f'(x)^2} dx$$

$$f(x)=x^2$$

$$df(x)=diff(x)$$

In MATLAB

```

>> syms x
>> f(x)=x^2;
>> L(x)=(1+(df(x))^2)^.5;
>> L1=L('1');
>> L3=L('3');
>> L13=L3-L1;
>> display(double(L13))
ans = 3.8467

```

Example 11: Average value of a function

Average value of a function $f(x)$ over the interval $[a,b]$ is given $1/(b-a) \int_a^b f(x) dx$. Find the average value of $f(x)=x^2+x$ over the interval $[1,3]$

In MATLAB

```

>> syms x
>> f(x)=x^2+x;
a =1;
>> b=3;
>> av(x)=1/(b-a)*int(f(x));
>> av13=av('3')-av('1')
av13 = 19/3

```

Reference:

Ginsburg, A., Calculus Problems and Solutions, Holden-Day, 1963

<http://www.sosmath.com/calculus/diff/der11/der11.html>

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